New Mathematics and Natural Computation
Vol. 12, No. 2 (2016) 157–173

DOI: 10.1142/S1793005716500113

Out-of-Equilibrium Dynamics with Heterogeneous Capital Goods

Vipin P. Veetil

Department of Economics
George Mason University
Mason Hall, 1st Floor, MSN 3G4
Fairfax, Virginia 22030, United States of America
vipin.veetil@gmail.com

This paper studies the problem of accumulating heterogeneous capital goods in an economy with imperfect markets populated by boundedly rational agents. It relaxes classical assumptions about information and cognition. The agents are not capable of computing an equilibrium path to steady state. Agents discover prices by interacting with each other. The economy accumulates a near-optimal mix of capital goods. The structure of interactions between agents filters their behavior in such a way that limited rationality at the micro-level does not translate to grossly inefficient outcomes at the macro-level.

Keywords: Heterogeneous capital; expectations; learning; equilibrium; price discovery.

1. Introduction

Capital goods are heterogeneous and specific.[1] It is costly to move capital goods from one use to another.[2] A ship cannot be instantaneously and costlessly transformed into an airplane. Such an economy faces the problem of accumulating an optimal mix of capital goods. The problem is aggravated by the lumpiness and irreversibility of capital investments.[3–7] We study the accumulation of capital goods in an economy with imperfect markets populated by boundedly rational agents. A Walrasian auctioneer does not clear the market for capital goods. Rather the prices of capital goods emerge from a decentralized process of interactions. Agents in our model are not capable of computing an equilibrium path to steady state. Agents form expectations about prices one period ahead, but do not form expectations about the time path of prices.

Despite the apparent imperfections, the economy accumulates a near-optimal mix of capital goods. As the exchange process unfolds, the differences in price expectation between agents declines, so does the forecasting errors. One period price forecasts inter-lock to produce a long-series of expectations that drive the economy toward steady state. Our model recovers near-classical results about economic dynamics without classical assumptions about the process of price determination or the rationality of economic agents.

Frank Hahn argued that a decentralized economy could accumulate an optimal mix of heterogeneous capital goods if and only if agents have rational expectations about the time path of prices.[8–10] Moreover, the initial price vector that asymptotically drives an economy towards equilibrium may not be unique.[11] Economic agents must be able to select one of the equilibrium paths to steady state and all agents must select the same path.

Since the 1970s, numerous economists have analyzed processes through which agents may form rational expectations.[12–15] These include minimizing least squared errors of price expectations and Bayesian updating. In recent years, economists have studied the problem of model uncertainty, where the model itself is a choice variable.[16,17] We extend the literature by studying economic dynamics in a world where agents do not have a model of the economy as a whole. Nor do the agents possess the information or cognition necessary to estimate parameters of the joint-distribution of state variables.

The paper is organized as follows. Section 2 presents the model. It describes the agents' rationality, information environment, and the market process. Section 3 presents the results. Section 4 extends the model by incorporating exogenous shocks to study the consequence of instability in the decision environment. Section 5 offers concluding remarks. Appendix presents the pseudo-code.

2. The Model

2.1. *Why an agent-based model?*

An agent-based model (ABM) is essentially an artificial economy *in silico*. ABM is useful to study systems that are difficult to characterize through analytical methods. Some systems cannot be fully described through equations, while others are difficult to study by manipulating equations.[18] ABM helps overcome the problem of mathematical intractability and thereby opens new vistas of problems to study using a formal language. An ABM has three parts.[19] The first part is a specification of individual agents. This includes description of their cognitive characteristics, access to information, and goals. All of which may vary across agents. The second part is a specification of the process of interactions between agents. And the third part is an environment through which or with which agents interact. We use an ABM because it is difficult to study out-of-equilibrium dynamics with analytical models. Our model code is freely available at: https://bitbucket.org/VipinVeetil/capital.

2.2. *Model setup*

There is one kind of labor and one kind of consumption good. There are I kinds of capital goods. An arbitrary capital good is denoted by $i \in \{1, \ldots, I\}$. Different kinds of capital goods have different productivity. The market for labor and the consumption good clear parametrically. Which means there is no excess demand for labor and consumption good. Stylized markets for labor and consumption good allow us to focus on the dynamics of the market for capital goods. The market for capital goods works through a binary matching procedure. The demand curve for the consumption good is given by Eq. (1)

$$g^d \equiv \frac{A}{p}, \tag{1}$$

where g^d is the demand for the consumption good, p is the price of the consumption good, and A is a parameter. The supply of labor l^s is fixed as given by Eq. (2).

$$l^s \equiv L. \tag{2}$$

There are two kinds of firms: capital goods producing firms and consumption good producing firms. Capital goods producing firms shall be refereed to as K-firms. Consumption good producing firms shall be refereed to as C-firms. K-firms produce capital using labor as input. K-firms do not use capital goods as inputs. C-firms produce the consumption good using capital goods as inputs. C-firms do not use labor as an input. Each K-firm produces one kind of capital good using the production function given by Eq. (3)

$$k = l^{\gamma}, \tag{3}$$

where k_i is the quantity of ith kind of capital, l is labor, and γ is a productivity parameter. Labor is equally productive in the production of all kinds of capital goods. All K-firms have the same production function.

C-firms produce using a Cobb–Douglas production function given by Eq. (4)

$$y = \prod_{i=1}^{I} k_i^{\theta_i}, \tag{4}$$

where θ_i is the Cobb–Douglas exponent of the ith kind of capital good. Different capital goods have different productivity as represented by different Cobb–Douglas exponents. The Cobb–Douglas form represents complementarities between different kinds of capital goods. All C-firms have the same production function. At the beginning of each time step, the K-firms place demand curves for labor. These demand curves are summed and matched with the supply of labor to determine wage. Each K-firm is allocated labor according to its demand. After which, K-firms produce capital goods and the market for capital goods opens. Each K-firm is matched with a C-firm. Each pair exhausts gains from trade. After which, C-firms produce the consumption good. The total output of the consumption good is matched with the demand for consumption good to determine its price for the period. The following subsections describe each part of the process in detail.

2.3. *The markets*

2.3.1. *Demand and supply of labor*

Each K-firm produces one kind of capital. J denotes the set of K-firms. $J_i \subset J$ denotes the set of firms producing capital good i. j denotes an arbitrary K-firm. Each firm $j \in J_i$ solves the profit maximization problem given by Eq. (5)

$$\begin{aligned} \max_{k_j, l_j} \quad & \pi_j = \mathbb{E}_j(q_i)k_j - wl_j \\ \text{subject to} \quad & k_j = l_j^\gamma, \end{aligned} \tag{5}$$

where $\mathbb{E}_j(q_i)$ is firm j's expectation of the price of capital of kind i, k_j is the quantity of capital firm j produces, and l_j is the quantity of labor firm j uses as input. The solution to the problem generates the demand curve for labor of firm j given by Eq. (6).

$$l_j = \left(\frac{\gamma\,\mathbb{E}_j(q_i)}{w}\right)^{\frac{1}{1-\gamma}}. \tag{6}$$

The demand for labor by all firms that produce the ith kind of capital is given by Eq. (7).

$$L_{J_i} = \sum_{j \in J_i} l_j. \tag{7}$$

The sum of the demand for labor by all firms is given by Eq. (8).

$$L^d = \sum_{J_i \subset J} L_{J_i}. \tag{8}$$

The total labor demand is set equal to the labor supply to compute the wage.

$$w^* = \frac{\gamma}{L^{1-\gamma}} \left(\sum_{J_i \subset J} \sum_{j \in J_i} \mathbb{E}_j(q_i)^{\frac{1}{1-\gamma}} \right)^{1-\gamma}. \tag{9}$$

2.3.2. *Capital goods market*

After receiving labor, K-firms produce output, and the capital goods market opens. Capital goods market does not have a Walrasian Auctioneer. Instead K-firms and C-firms meet through a process of binary matching. Each K-firm is matched with a C-firm. Each pair exhausts gains from trade using the following procedure. K-firm submits an ask-price and C-firm submits a bid-price for a unit of a capital good. K-firm's ask-price equals the average cost of production. C-firm's bid-price equals the life time incremental revenue product of a unit of capital. If the ask-price is less than the bid-price, the C-firm and the K-firm trade a unit of the capital good. The trade price is the mean of bid-price and ask-price. The K-firm makes offers and the C-firm makes bids until K-firm exhausts inventory or the offer-price is greater than the bid-price.

2.3.3. *Demand and supply of the consumption good*

Once the capital goods market closes, C-firms produce the consumption good. Total output is matched with demand to determine the price of the consumption good, which is given by Eq. (10).

$$p^* = \frac{A}{g^s}. \tag{10}$$

Price information is passed to C-firms. This completes a time step.

2.4. *The firms*

2.4.1. *K-firms*

At the beginning of each time step, K-firms compute their demand curve for labor based on the expected price of capital goods. The demand curve of individual K-firms is given by Eq. (6). The expected price of capital good, denoted by $\mathbb{E}_j(q_i)$, is an argument of the demand curve. The subscript j indicates each firm's expectation of price of the ith kind of capital good may be different. K-firms do not know the price at which their capital will sell before capital markets open. Each K-firm forms price expectations based on its past trades as given by Eq. (11).

$$\mathbb{E}_{j\in J_i,t}(q_i) = f(\mathbf{q}_{j\in J_i,t-s}), \quad \text{with } s \in \{1,\ldots,S\} \tag{11}$$

$\mathbf{q}$ is a vector of the prices at which capital was traded in the last time step. Each unit of capital traded between K-firm and C-firm has a different prices. In the model, s is 1 and f is the arithmetic mean. The expected price of a capital good for a K-firm for period t is the mean price at which the firm sold the capital good in period $t-1$. Different K-firms form different price expectations based on different trade histories. K-firms do not know each other's price expectations, nor do they estimate parameters of distribution of prices to predict future prices.

K-firms which have unsold inventory from period $t-1$ do not buy labor in period t. K-firms with unsold inventory reduce the ask-price of capital by fixed percent x. In the first time step, K-firms do not have a history of past trades to form price expectations. K-firms begin with an expected price that is a random draw from a *uniform* (1,10) distribution.

2.4.2. *C-firms*

When a C-firm is matched with a K-firm, the C-firm offers a bid-price for a unit of capital. The bid-price is the maximum price the C-firm is willing to pay for a unit of capital. The bid-price equals the life time incremental revenue product of the unit of capital. The incremental revenue product (IRP) is the discrete unit equivalent of marginal revenue product. The life time incremental revenue product ($LIRP$) is the sum of the IRP for all periods to infinity.

The IRP in period t of firm j for capital kind i bought in period t is given by Eq. (12).

$$IRP_{j,t} = \mathbb{E}_{j,t}(p)\left\{\left[(k_{i,t}+1)^{\theta_i} \times \prod_{s=1,s\neq i}^{I} k_{s,t}^{\theta_s}\right] - \prod_{s=1}^{I} k_{s,t}^{\theta_s}\right\}, \tag{12}$$

where $\mathbb{E}_j(p)$ is firm j's expectation of the price of the consumption good in period t. The IRP in period $t+1$ of firm j for capital kind i bought in period t is given by (13).

$$\begin{aligned} IRP_{j,t+1} =& \mathbb{E}_{j,t}(p)\left(\left(((1-\delta)(k_{i,t}+1))^{\theta_i} \times \prod_{s=1,s\neq i}^{I} ((1-\delta)k_{s,t})^{\theta_s}\right)\right. \\ &\left. - \prod_{s=1}^{I} ((1-\delta)k_{s,t})^{\theta_s}\right), \end{aligned} \tag{13}$$

where δ is the rate of depreciation of capital goods. Equation (13) can be simplified using the assumption that $\sum_{i=1}^{I} \theta^i = 1$.

$$IRP_{j,t+1} = (1-\delta)\mathbb{E}_{j,t}(p)\left(\left((k_{i,t}+1)^{\theta_i} \times \prod_{s=1,s\neq i}^{I} k_{s,t}^{\theta_s}\right) - \prod_{s=1}^{I} k_{s,t}^{\theta_s}\right). \tag{14}$$

Lifetime incremental revenue product (LIRP) is the sum of IRP from the current period to infinity as given by Eq. (15).

$$LIRP_{j,t} = \sum_{i=0}^{\infty} (1-\delta)^i IRP_{j,t}. \tag{15}$$

Equation (15) simplifies to Eq. (16) because $0 < (1-\delta) < 1$.

$$LIRP_{j,t} = \frac{IRP_{j,t}}{\delta}. \tag{16}$$

2.5. *Agents' rationality*

Neither C-firms, nor K-firms compute the time path of prices. C-firms compute incremental revenue product. However, when forming bid-prices, C-firms do not consider the impact of potential future trades on the incremental product of capital goods. When bidding for capital goods, C-firms expect the price of the consumption good in period t to be the same as the realized price in period $t-1$. When placing demand for labor, the K-firms expect the price of capital goods in period t to be the same as the realized price in period $t-1$. The agents do not identify trends in the time-series of prices.

2.6. *Information environment*

C-firms and K-firms do not have a mountain top view of the economy. C-firms and K-firms update beliefs as the trading process unfolds. The K-firms begin with random

Table 1. Parameters.

Parameters	Description	Bound	Baseline values
A	Goods demand	$\mathbb{R}^{++}$	1000
γ	Cobb Douglas exponents of the ith kind of capital	$(0, 1)$	0.5
θ_i	Productivity of labor in producing capital	$\mathbb{R}^{++}$	$\theta_1 = 0.25$ and $\theta_2 = 0.5$
m	Number of capital firms of each kind of capital	$\mathbb{I}^{++}$	1000
n	Number of kinds of capital	$\mathbb{I}^{++}$	2
δ	Depreciation	$(0, 1)$	0.1
L	Labor supply	$\mathbb{I}^{++}$	100,000
x	Unsold capital price adjustment	$(0, 1)$	0.1

prices and update their price expectations as the trading process unfolds. The C-firms decide how much capital to buy based on their beliefs about the price of the consumption good. They update beliefs as the market process unfolds. Both C-firms and K-firms commit errors because they make decisions using simple rules and with limited information. The exchange process provides opportunities for correcting these mistakes.

2.7. *Parameters*

Table 1 lists parameters of the model. Column 1 provides the symbols of parameters. Column 2 provides a description of parameters. Column 3 provides the theoretical bounds of parameters. $\mathbb{I}^{++}$ stands for strictly positive integers and $\mathbb{R}^{++}$ stands for strictly positive real numbers. Column 4 provides the baseline values of parameters in the the model.

3. Results

Result 1. *The economy accumulates a near-optimal mix of capital goods and converges to the neighborhood of steady state.*

Result 2. *Price expectations converge and errors in price expectations decline as the trading process unfolds.*

Evidence and Discussion: Figure 1 plots the relation between the ratio of stock of capital goods and the ratio of Cobb–Douglas exponents of capital goods. The Cobb–Douglas exponents represent the productivity of the two kinds of capital goods. The capital stock ratio varies monotonically with the productivity of the capital goods. The economy accumulates more of the more productive capital good. Figure 2 plots the relation between ratio of price of capital goods and the ratio of Cobb–Douglas exponents of the capital goods. The price of the more productive capital is greater than the price of the less productive capital. Figure 3 plots the errors in price expectation of the two capital goods of K-firms. Figure 4 plots the variations in the price expectation of the two capital goods among K-firms. The errors in price expectations and the variations in price expectations declines as the exchange process unfolds. One of the

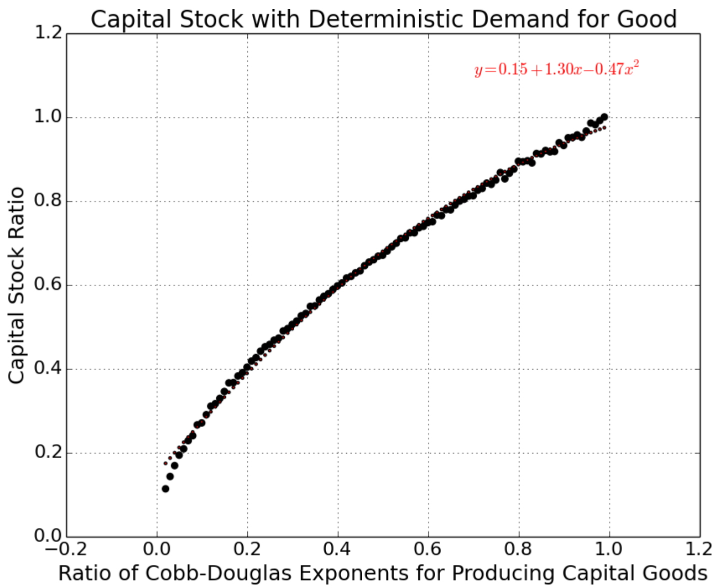

Fig. 1. Capital stock ratio.

reasons for the rapid learning in the model may be the fact that the market for consumption good and market for labor clear parametrically. The economy goes to the neighborhood of steady state, but does not reach steady state as the errors in price expectation and variations in price expectation do not decline to zero.

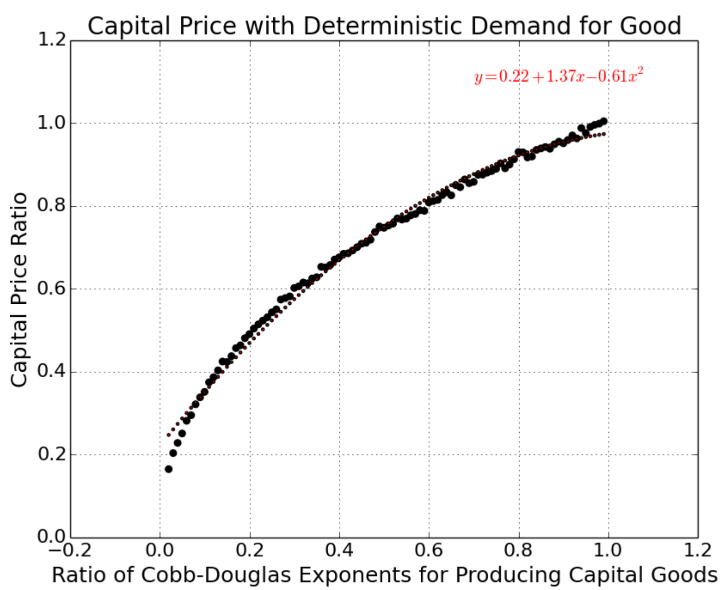

Fig. 2. Capital price ratio.

Fig. 3. Errors in capital price expectation.

The data for Figs. 1–4 are collected in the following manner. Figures 1 and 2 are created using data generated by running the economy forward in time for 1000 time steps, for 100 different ratios of Cobb–Douglas exponent. Figure 1 plots the mean capital ratio of the last 100 time steps. Figure 2 plots the mean capital price ratio of the last 100 time steps.

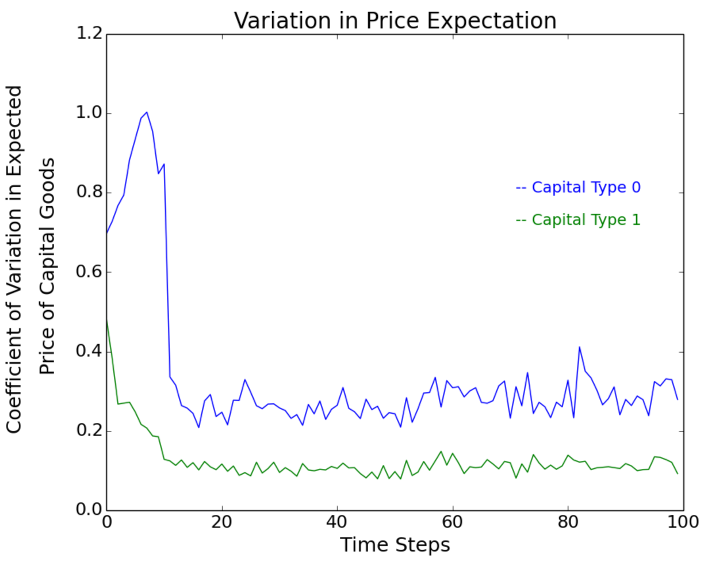

Fig. 4. Variation in capital price expectation.

Figures 3 and 4 plot the errors in capital price expectation and variation in capital price expectation of the first 100 time steps, with the ratio of Cobb–Douglas exponents fixed at 0.2. Each K-firm computes the root mean squared error in its expectation of capital price. Figure 3 plots the mean of the root mean squared errors of all K-firms. Figure 4 plots the coefficient of variation in expected price of the capital goods among the K-firms.

4. Unstable Decision Environment

Previous sections illustrate that agents with limited computational ability gradually learn near-optimal behavior. However, the learning happens within a stable environment. There are no exogenous shocks to the system. In this section, we introduce stochastic demand for the consumption good to study learning in an unstable decision environment. The parameter A of the demand for consumption good follows a Geometric Brownian Motion, with zero drift. The initial value of the motion is the deterministic value of A. Figure 5 plots the relation between the ratio of stock of capital goods and the ratio of Cobb–Douglas exponents of capital goods, when demand for the consumption good is stochastic. Figure 6 plots the relation between ratio of price of capital goods and the ratio of Cobb–Douglas exponents of the capital goods, when demand for the consumption good is stochastic. Comparing Fig. 1 to Fig. 5 and Fig. 2 to Fig. 6 shows stochastic demand for consumption good does not alter the principal result of the last section. Even when the demand for consumption good is stochastic, the economy accumulates a near-optimal mix of capital goods.

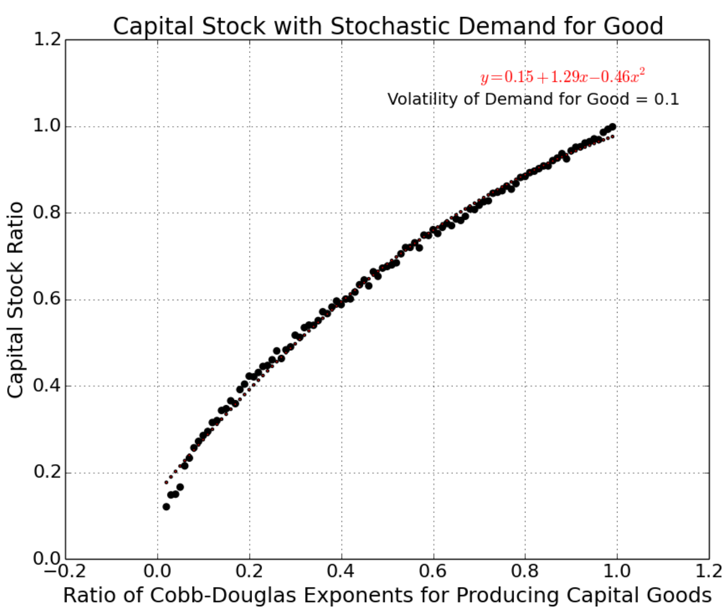

Fig. 5. Price and stock of capital over time with stochasticity.

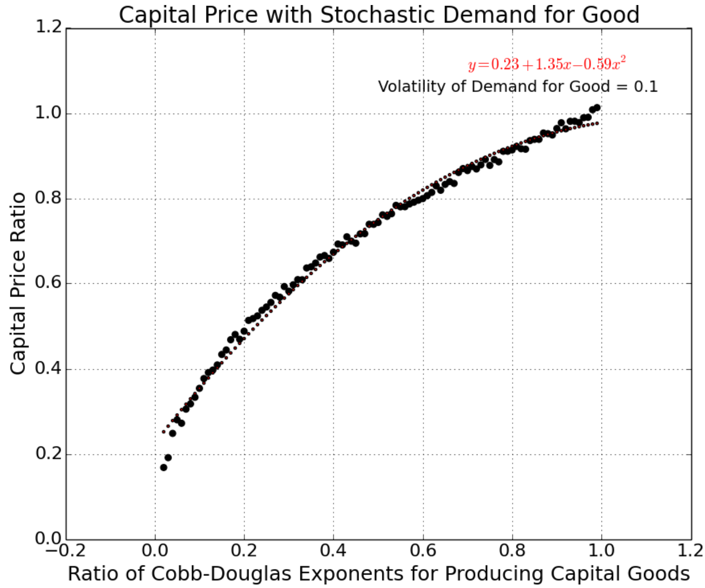

Fig. 6. Errors and variation in price expectation over time with stochasticity.

The stochasticity of the demand for consumption good does however affect the variability of the capital stock ratio and variability of capital price ratio. Figure 7 plots the relation between the volatility of the Geometric Brownian Motion of parameter A and the coefficient of variation of capital stock ratio. Figure 8 plots the

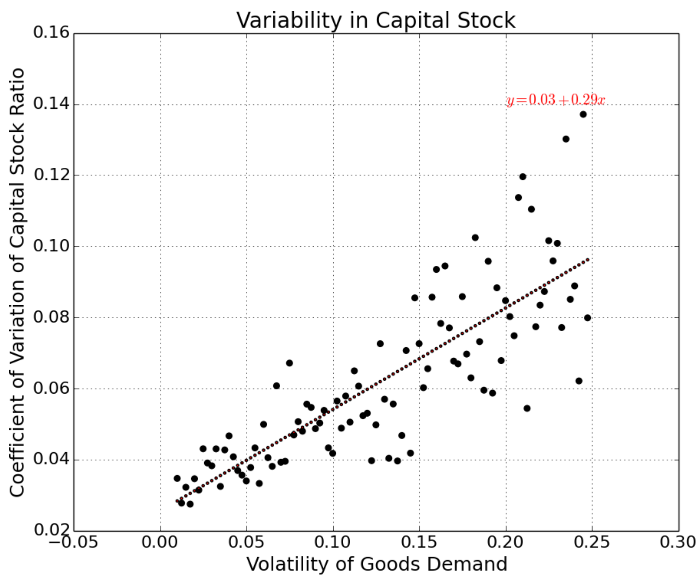

Fig. 7. Variance in price and capital stock ratio with stochasticity.

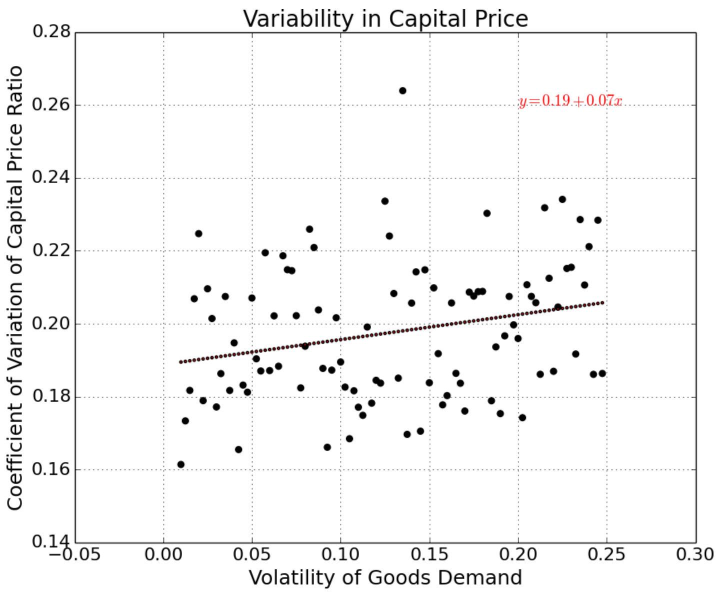

Fig. 8. Variance in price expectations with stochasticity.

relation the volatility of the Geometric Brownian Motion of parameter A and the coefficient of variation of capital price ratio. Like in the previous section, the economy is run forward for 1000 time steps for 100 different values of volatility. The coefficient of variation in capital stock ratio and the coefficient of variation in capital price ratio is computed with data from the last 100 time steps. Figure 7 shows the variability of the capital stock ratio increases with the stochasticity of consumption demand. Figure 8 shows the variability of capital price ratio increases with the stochasticity of the demand for consumption good.

Why does the variability of capital price ratio increase with stochasticity of consumption good demand? In other words, why does the differences in opinion among agents increase in response to a common shock? After all, variable A determines aggregate demand, shocks to A are not idiosyncratic shocks. The increase in the differences in expectation among agents happens in the following manner. A demand shock changes the incremental revenue product of a unit of capital for C-firms. However, the change is not uniform across firms. The change in the incremental revenue product of a C-firm depends on its capital stock. The incremental revenue product of C-firms do not change uniformly because different C-firms have different stocks of capital. The bid-price of a C-firm is the life time incremental revenue product. The bid-prices of different C-firms changes differently because their incremental revenue product changes differently. The trade-price is the mean of the bid-price and ask-price. The trade-prices of capital changes differently because bid-price changes differently. K-firms use trade-prices of capital to form price expectations. The price expectations of K-firms change differently because they meet different C-firms, whose bid-prices change differently. *In imperfect markets with boundedly rational agents, aggregate shocks can have consequences similar to that of idiosyncratic shocks.*

5. Concluding Remarks

Frank Hahn argued that two conditions are necessary for an economy with heterogeneous capital goods to traverse an equilibrium path to steady state.[8] One condition is all agents must have the same expectations about the time path of relative prices. The other condition is all agents must be right. Our paper studies an economy with durable capital goods, imperfect markets, and boundedly rational agents. We find that near-homogeneous and near-correct expectations can be a consequence of the market process rather than a pre-requisite for convergence to equilibrium. A decentralized process of decision-making can produce the conditions necessary for its stability. Agents learn by interacting with each other and adjusting prices in response to forecasting errors. Agents correct for decisions based on incorrect price expectations by accumulating one capital good and depreciating the other capital good. One period expectations interlock to produce a long series of expectations that drive the economy towards equilibrium.

Hahn was right in saying that in a system with "durable concrete objects, the path of the system will at any time be strewn with remnants of past mistakes".[20] But the consequences of past mistakes depend on the trading process. Some trading processes erode the remnants of past folly, while others create the conditions for yesterday's mistakes to become tomorrow's crises. We study a trading process which erodes the consequences of past errors. Nothing said in this paper suggests real world markets are stable. As Schumpeter warned, one must not fall prey to the Ricardian Vice.[21] As to whether real world trading processes dampen or aggravate past errors is an empirical question, which is not investigated in this paper.

The results of our model ought to be interpreted with caution. The model merely illustrates convergence to equilibrium in a production economy with two capital goods, with stylized markets for labor and consumption good. We do not investigate the problems of dependence between multiple markets which are out-of-equilibrium or the problems associated with the use of money. Though narrow, our paper does poke at a beehive central to macroeconomic theory: the relation between expectations and durable capital goods.[22] In a 1937 lecture Keynes said that if he were to rewrite the General Theory, he would begin with the assumption that short period expectations were always fulfilled and then show what difference it makes when short-period expectations are disappointed.[23] The problem of failed expectations can be particularly acute in a capital using economy. Hayek was more optimistic than Keynes about the capability of market interactions to produce reasonable expectations in a capital using economy.[24] Hayek thought, "It is precisely through the disappointment of expectations that a high degree of agreement of expectations is brought about".[25] Unfortunately, there are few if any formal results on how the interplay between expectations and capital influence the stability of economic systems.

Nearly one and half centuries after Walrass formulation of General Equilibrium, economists have little to say about the stability of general equilibrium in a production economy with interacting agents.[26] Much of the work on stability of general equilibrium studies systems without agent-interactions.[27–29] This is a testimony to the difficulty of the problem.[30] The mathematics of the 19th century does not lend itself to the study of systems with distributed decision-making.[31,32] Agent-based modeling is a way forward in studying some of the hard problems of economic theory.[33] The stability of a production economy is prime candidate. The stability of the system is likely to depend on expectation formation rules,[34] relation between capital goods,[22] time lag between investment and production,[35] market micro-structure,[36] and the topology of agent-interactions.[37] Much work remains to be done.

Despite the hardness of the aforementioned problems, they merely scratch the surface of real world dynamics. More capital is seldom more of the old capital. Japanese today do not own more horse-buggies than their great-grand parents. Novel capital goods produce evolutionary dynamics with the destruction of some economic relations and the creation of new goods, new markets, and new ways of doing things.[38,39] The stability of evolutionary dynamics in a production economy with capital goods remains unchartered territory.

Acknowledgments

I thank Robert Axtell, Peter Boettke, Gianfranco Giulioni, Akshaya Vijayalakshmi, Richard Wagner, Lawrence White, members of the Graduate Student Paper Workshop at George Mason University and two anonymous referees. All errors are my own.

Appendix A. Pseudo Code

```
PROCEDURE Initialization
   Create capital goods firms and consumption goods firms
END

PROCEDURE Simulation Step * (this procedure is run repeatedly in the simulation
loop)
   RUN Labor-Demand-Supply
   RUN Capital-Goods-Market
   RUN Consumption-Good-Demand-Supply
END

PROCEDURE Labor-Demand-Supply
   Ask capital goods firms that do not have inventory to submit labor-demand
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```
  Sum labor demand
  Match aggregate labor demand to supply to compute wage
END

PROCEDURE Capital-Market
  Match each capital goods firm with a consumption good firm
  RUN Pairwise-Trade
END

PROCEDURE Pairwise-Trade (* this procedure is run for all pairs of firms)
  Select a pair of firms
  For each pair:
      RUN Exhaust-Gains-From-Trade
END

PROCEDURE Exhaust-Gains-From-Trade (* this procedure is run in a loop for each
pair)
  Ask the capital goods firm to form an ask price
  Ask the consumption good firm to form a bid price
  If: bid>ask and capital goods firm has inventory, allow trade;
    Ask both firms to store price information.
  Else: exit loop

PROCEDURE Consumption-Good-Demand-Supply
  Ask consumption good firms to produce
  Collect total output and match with demand curve for consumption good
  Pass price information to consumption good firms
END
```

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