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The Complexity of Coordination

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The traditional mechanism of finding Nash equilibria presumes economic actors are capable of performing computations that even computers would take far too long to complete. A decentralized and parallel process of interactions between simple economic actors is presented as a more plausible microfoundation of the concept of Nash equilibria. It is found that agent interactions on a scale-free network converge to an equilibrium within reasonable time. *NP* computational complexity of Nash equilibria does not diminish its empirical relevance.
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INTRODUCTION

Nash equilibrium is the central solution concept of games in economics and other behavioral sciences. And for good reasons, it is intuitively appealing because it describes a constellations of strategies from which unilateral deviations do not pay. It is empirically significant because it describes the behavior of autonomous actors pursuing their own ends. There are however three problems on which the concept is silent. The first problem is that many games have multiple Nash equilibria and players may not know which equilibrium to select [Maskin 2011]. This problem is studied by the literature on 'refinement of equilibria' [Myerson 1978; Kohlberg 1990; Van Damme 2012]. The second problem is that in many games the cognition and information necessary for players to deduce Nash equilibria is simply unrealistic. This problem has been studied by the literature on games between boundedly rational agents [Simon 1982; Rubinstein 1998]. The third problem is that the concept of Nash equilibria does not say anything about how long a system might take to find it. Traditionally, it is presumed that agents can deduce the Nash equilibria given information about the game and other agents. Recent results on the computational complexity of Nash equilibria suggest that this is an unrealistic presumption [Goldberg and Papadimitriou 2006]. Computing Nash equilibria is a hard problem; in some circumstances it may take far too long for the outcome to be useful.

A different approach to finding Nash equilibrium is agent interaction. This paper investigates the conditions under which interaction between large numbers of autonomous agents can discover a Nash equilibrium. Each agent follows simple rules of behavior and has access to local information, but does not know the variables that describe the state of the system. There is no leader, sanction, tacit knowledge of a focal point, or commonly observed variable on which to condition strategy. No appeal is made to backward induction, iterated dominance, invariance principle, or any other mechanism to select an equilibrium from the set of equilibria. Can agents who follow simple rules and do not know much about the strategies of other agents find a Nash equilibrium? Can we recover the concept of Nash equilibrium with agents who cannot compute it? To answer these questions, the transient dynamics of parallel and decentralized interactions on a scale-free network is studied.[1] Computational experiments show that agent interactions discover Nash equilibrium in polynomial time on a scale-free network. It is hypothesized that the power–law degree distribution of scale-free networks produces *cascades to equilibrium*. This paper is a small step toward building computational microfoundations of a central concept for solving economic games.[2]

The problem of how a system of autonomous interacting agents might find Nash equilibria has been studied using laboratory experiments [Cooper et al. 1990; Van Huyck and Beil 1990]. A variety of results suggest that pre-play negotiation [Alm and McKee 2004; Blume and Ortmann 2007], information about the strategies of other players [Berninghaus and Ehrhart 2001], presence of a leader [Brandts and Fatas 2007], shared conceptions of focal points [Mehta and Sugden 1994] all help in finding equilibrium. Field studies show that public rituals [Chwe 2013], sanctions [Fehr and Fischbacher 2004; Kitts 2006], leadership [Hermalin 1998], and group attributes play a role in how successful a group is in finding a desirable Nash equilibria.

While laboratory experiments and field studies are useful in understanding the empirical dynamics of group behavior, they are less effective in studying the time-dimension of transient behavior. How long would it take for an economy to traverse from one equilibrium to another in response to a change in monetary policy? How long would it take for a society to find the new equilibrium in response to a change in retirement policy [Axtell and Epstein 1999; Axtell 2001; Axtell and Epstein 2010]? Ostrom [2000] found that under some circumstance social norms emerged to solve collective actions problems. How long does it take for a community to develop such norms? Such questions involve the interaction between thousands—if not millions—of economic actors on complex network topologies. It is costly, in some cases prohibitively so, to study transient dynamics using laboratory experiments. Agent-based models are useful to study such problems. An agent-based model is a population of agents along with rules of interaction. Each agent is endowed with some goals, rules of behavior, and ways of accessing information. The artificial economy is run forward in time, and the data that emerge from agent interactions are collected for inference [Axelrod 1997]. Agent-based models are particularly useful to study problems that involve too few variables to use statistical-mechanics and too many variables to use nineteenth century mathematics [Weaver 1948]. Agent-based modeling is a new mathematics for social sciences [Borrill and Tesfatsion 2011].

"Complexity of Computing Nash Equilibria" section summarizes the recent results on the computational complexity of the agent-deduction mechanism of computing Nash equilibria. "Decentralized Interactions" section presents a decentralized and parallel interaction mechanism. "Complexity of Decentralized Interactions" section studies the complexity of the interaction mechanism on scale-free network using an agent-based computational model. "Concluding Remarks" section offers concluding remarks. Appendix A presents the pseudoCode and Appendix B lists the parameters of the model.

COMPLEXITY OF COMPUTING NASH EQUILIBRIA

Nash [1951] proved the existence of fixed points in noncooperative games, i.e., games in which agents make independent decisions without the help of an external authority to enforce contracts. However, Nash's proof was not constructive, i.e., it did not describe a way of finding equilibrium. Nash's work left two open questions. One, what are the mechanisms by which equilibrium can be found? Two, how do these mechanisms compare in terms of the time necessary to find equilibrium? Without a mechanism to find equilibrium within reasonable time, there is little reason to believe that equilibrium will ever be observed. In other words, the empirical significance of Nash's proof depends on the existence of plausible discovery mechanisms.

Traditionally, it has been thought that each agent can deduce Nash equilibria using information about other players and the game itself. This mechanism can be formalized using a Universal Turing Machine (UTM). A UTM is characterized by an input tape, a work tape, and an output tape [Arora and Barak 2009]. The details of the problem to be solved are inscribed on the input tape; intermediate steps in the computation are stored on the work tape; and the solution is printed on the output tape. A UTM is a bare bones description of a computer. Time complexity measures the number of steps necessary for a UTM to solve a problem as a function of the size of inputs. A problem is in class-*P* if this function is a polynomial. These are the class of problems that can be solved by computers in a realistic amount of time. School-book long division belongs to class-*P*; it has a complexity of $O(n^2)$; the steps necessary to solve the problem increase by the square of the size of the problem. Class-*NP* contains problems for which no known polynomial time algorithm exists, but whose answers can be checked in polynomial time.[3] A *NP* problem is a bit like finding a needle in a haystack: it is very hard to find a needle, but easy to confirm that a needle has been found. A prototypical example of a problem in class-*NP* is the problem of finding a clique in a graph. No known polynomial time algorithm exists to solve the clique problem, i.e., no algorithm is known for which the time to solve the problem grows as a polynomial of the size of the graph. However, once a clique is found it can be checked in polynomial time. A problem is *NP*-hard if it belongs to class-*NP* and is at least as hard as any other problem in the class. A problem is *NP*-complete if it belongs to class-*NP* and is *NP*-hard.

Recent results suggest that for nearly all problems of interest, agent-deduction would take far too long to compute Nash equilibria [Goldberg and Papadimitriou 2006; Etessami and Yannakakis 2010], i.e., as the number of agents and or the number of equilibria increase, the time steps necessary to solve the problem increase more than polynomially. The problem of finding Nash equilibria is *PPAD*-complete for a large class of multiplayer games, where $PPAD \subset TFNP \subset FNP \subset NP$[4] [Daskalakis and Papadimitriou 2006, 2009]. Restricting the domain of search or the number of players does not reduce the complexity of the problem. Finding Nash equilibria that satisfy a given payoff is in class-*NP* [Conitzer and Sandholm 2003, 2008]. Computing Nash equilibria in a game with just two players is *PPAD*-complete [Chen and Deng 2006]. These results extend to simple stochastic games with multiple players [Ummels and Wojtczak 2009].

Some mathematical problems are easier to solve when formulated as a infinite horizon problem than as a finite horizon problem, for instance Bellman equations help solve infinite horizon optimization problems by exploiting their recursive structure. However, the infinitely repeated structure of a game does not help simplify the process of finding Nash equilibria [Ummels 2008]. The problem of finding equilibria for infinitely repeated games is as difficult as that of finding it for a finite horizon game [Borgs et al. 2008].

There are few results on the complexity of games on networks. They too point toward the infeasibility of the traditional mechanism of finding Nash equilibria. The problem of computing pure strategy Nash equilibria in *congestion games* is *PLS*-complete for directed and undirected networks, where $PLS \subset TFNP \subset FNP \subset NP$[5] [Fabrikant and Talwar 2004; Ackermann and Vöcking 2008]. It can however be computed in polynomial time for symmetric networks. The problem of determining whether a game on a d-dimensional grid has a pure strategy Nash equilibrium is *NEXP*-complete when $d \geq 2$, where $NP \subset EXP \subset NEXP$ [Daskalakis and Papadimitriou 2005]. It is *NP*-hard to compute the best and worst pure strategy Nash equilibria in selfish routing games on a network [Fotakis et al. 2002].

Because the time necessary to find Nash equilibria using the conventional mechanism increases more than polynomially, it is not plausible that this mechanism is at work in the real world. Proposition 1 summarizes the results on complexity of computing Nash equilibria.[6]

Proposition 1 *The agent-deduction mechanism of computing Nash equilibria is sufficiently difficult that it is simply not a plausible description of real world dynamics.*

There are two ways to respond to Proposition 1. The first is to ignore it. The second is to look for other plausible mechanisms of computing Nash equilibria. This paper takes the second approach. Are there decentralized mechanisms of computing Nash equilibrium? If yes, what is their computational complexity? "Decentralized Interactions" section describes a decentralized mechanism and "Complexity of Decentralized Interactions" section presents results on the complexity of the mechanism on a scale-free network.

DECENTRALIZED INTERACTIONS

There are a set of agents $A = \{1, ..., A\}$ and a set of states $S = \{1, ..., S\}$. Denote an arbitrary agent by a, an arbitrary state by s, and an arbitrary time by t. Interactions between agents occur at a set of times $T = \{1, ..., T\}$. Each agent possesses an information set $I^a(t)$ at each time $t \in T$; $I^a(0)$ is agent a's initial information set. The information set of an agent is the states of its neighbors; $I^a(t) \in \mathbb{N}^n$ if agent a has n neighbors. This information set is updated only when an agent is activated. At every time step, some proportion of agents are activated. An activated agent collects information from its neighbors and uses this information to change its state; let $F : I \to s \in S$ be the function that maps information onto a state. Suppose $F = mode(I)$, i.e., each agent changes its state to that which is most common among its neighbors; in case of tie, an agent chooses a random element in the set. Suppose D is the mean of the degree distribution of the network of agents, i.e., each agent on average has D neighbors. The interaction process is given by a history parameterized mapping of the information sets $G_I : \mathbb{N}^{AD} \to \mathbb{N}^{AD}$, or as a history parameterized mapping of the states of the agents $G_S : \mathbb{N}^A \to \mathbb{N}^A$.

$$I(t+1) = G_I[I(t)] \tag{1}$$

$$S(t+1) = G_S[S(t)] \tag{2}$$

An equilibrium is a state in which all agents have the same information set $I^* = G_I(I^*)$, or equivalently the same states $S^* = G_S[S^*]$. The definition of F as mode guarantees the existence of equilibria, in fact, it guarantees the existence of S equilibria.

Suppose each agent begins with a random state, would the above decentralized process converge to an equilibrium? The stability of the process cannot be proved by constructing a Lyapunov function,[7] the process is not necessarily monotonic, nor is there any reason to presume that the process necessarily converges to an equilibrium. This paper studies the process by simulating agent interactions. The stability of the process is studied on a scale-free network. Scale-free networks have asymptotic power–law degree distribution; the Barabási and Albert [1999] algorithm is used to build a scale-free network. A barebones version of the model code is available at https://bitbucket.org/VipinVeetil/network_coordination. A comprehensive version of the model code is available at https://github.com/DavoudTaghawiNejad/The-Network-Origins-of-Coordination.

COMPLEXITY OF DECENTRALIZED INTERACTIONS

Result 1 *Agent interactions on a scale-free network find equilibrium in polynomial time.*

Figures 1 and 2 gives the number of time steps necessary to equilibrate an interacting population of agents on a *scale-free network*. The interaction process is terminated once 99.9 percent of the agents have the same state. Figure 1 gives the number of time steps necessary for varying numbers of agents. It shows that as the number of agents increase, the time steps to convergence increase sublinearly. The relationship is fitted using a second-degree polynomial: $y = -1.9 \times 10^{-6}x^2 + 0.515x + 6292$. It is hypothesized that the power–law degree distribution of scale-free networks is the reason behind these phenomena. The existence of a few agents who are highly connected and many agents who are scarcely connected creates the conditions for rapid spread of a state when some highly connected agents come to adopt it. The power–law degree distributions present the conditions that enable cascades to equilibrium, as can be seen on this video: https://youtu.be/ilZa4saEMAQ. A wide variety of real world social networks are scale-free [Barabási et al. 2009]. The preponderance of scale-free networks may have something to do with their ability to create consensus within reasonable time.

Figure 2 gives the number of time steps necessary for varying numbers of equilibria. It shows that increasing the number of equilibria does not impact the time to convergence. Furthermore, increasing the number of agents merely increases the number of time steps necessary for convergence, but does not change the relation of dependence between number of equilibria and time to convergence.

Figure 3 shows one run of the scale-free network with three states. It shows the out-of-equilibrium dynamics of the network. Figure 4 presents box-plots of time to convergence of the scale-free network with different means of the degree distribution. Each box-plot is made with data from 1000 runs. The mean and variance of time to convergence decreases

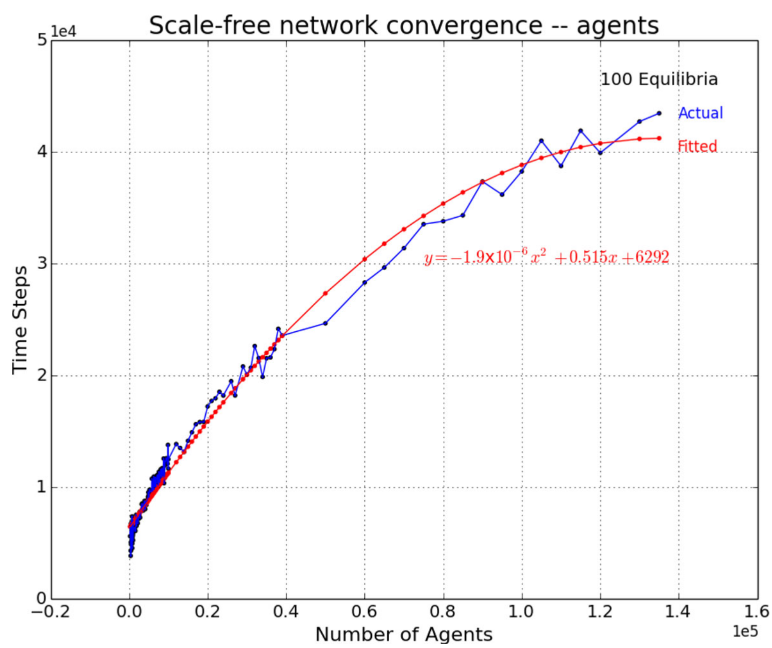

Figure 1. Scale-free network convergence with varying numbers of agents.

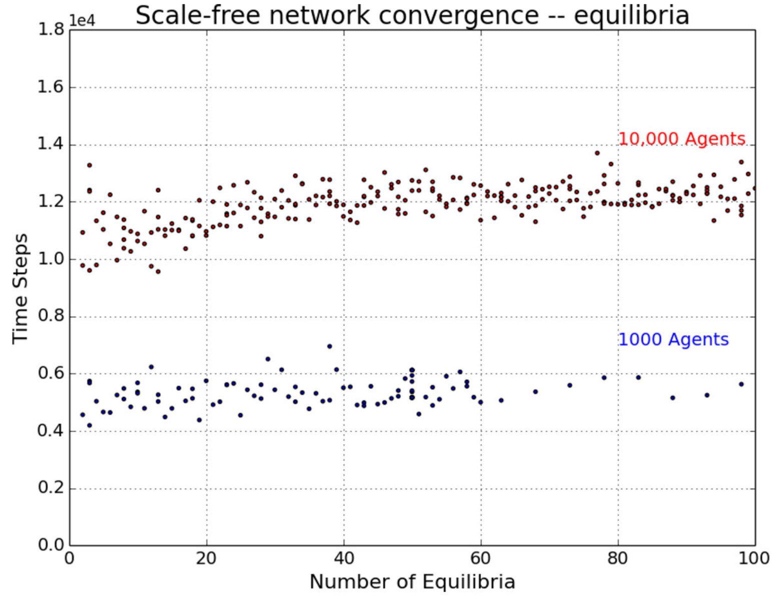

Figure 2. Scale-free network convergence with varying numbers of equilibria.

as the mean of the degree distribution increases. A more connected scale-free network converges to equilibrium in lesser time steps than a less-connected network.

These results suggest that the empirical relevance of the concept of Nash equilibria is not diminished by the recent results on the computational complexity of Nash equilibria. Decentralized and parallel interactions between economic actors can in certain circumstance discover equilibrium within reasonable time. In fact, such interactive-processes are a more plausible description of real world dynamics than agent-computation. Heroic assumptions about agent-rationality are not necessary for the concept of Nash equilibria to be meaningful in understanding social behavior.

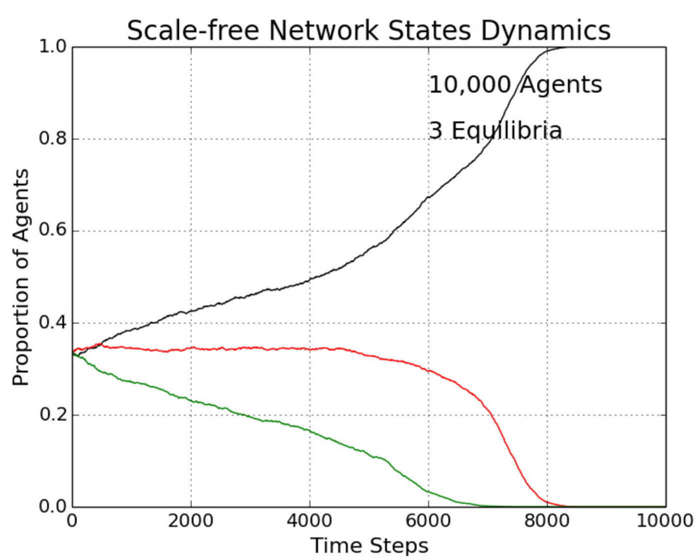

Figure 3. Out-of-equilibrium dynamics of scale-free network.

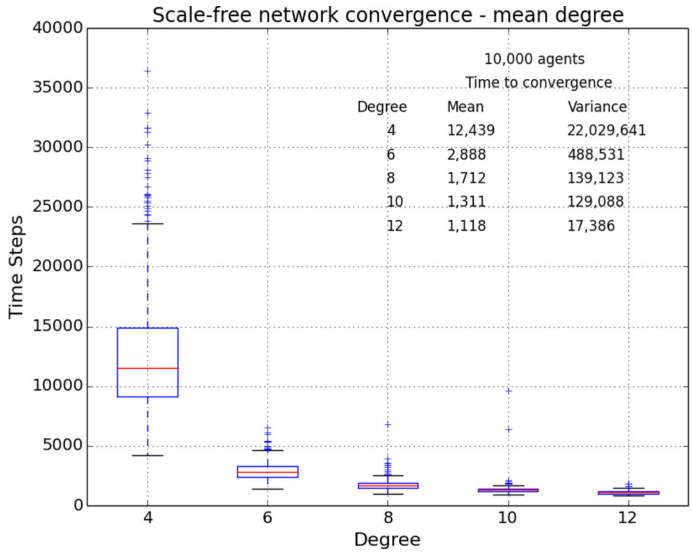

Figure 4. Scale-free network convergence with varying means of degree distribution.

CONCLUDING REMARKS

Imagine a large social system with people who have different beliefs. Suppose that at every time step, some proportion of the individuals imitate their neighbors by adopting the most popular belief in their locale. Would the system converge to an equilibrium? Traditionally, such problems have been solved using pre-play talk [Farrell and Rabin 1996], a focal point [Shelling 1960; Sugden 1995], or a commonly observed variable on which to condition strategy. This paper presents an alternate solution: agent interactions. The interaction between simple agents is studied on a scale-free network. Agent interactions on a scale-free network find a Nash equilibrium within reasonable time. The system finds equilibrium despite the fact that no agent models the behavior of other agents, or knows the state variables that describe the evolution of the system as a whole.

Our results challenge the idea that the concept of Nash equilibria is not useful because a computer cannot compute it within reasonable time. Some have argued that no rational player is more powerful than a computer. So if a computer cannot do it, nor can a system with players less powerful than a computer [Daskalakis and Papadimitriou 2009]. An economy however operates very differently from a computer. It is a system of decentralized and parallel interactions. This paper shows that interactions between agents can lead a system as a whole to converge to an equilibrium, though no agent is *computing* a Nash equilibrium. In an economy, computation happens not only within the minds of agents, but also through the process of interactions between agents. This insight is more than a century old. In the Manual of Political Economy, Vilfredo Pareto notes how the problem of solving systems of equations to derive equilibrium becomes increasingly difficult as the number of equations increases, so much so that it becomes impossible for all practical purposes. Pareto goes on to say:

> ...it would no longer be mathematics which would come to the aid of political economy, but political economy which would come to the aid of mathematics. In other words, if all these equations were actually known, the only means of solving them would be to observe the actual solution which the market gives[8] [Pareto 1971, 171].

In a recent paper summarizing the literature on the computational complexity of Nash equilibria, Daskalakis and Papadimitriou [2009] argue that the results on hardness of computing Nash equilibria prove that the concept of Nash equilibria is not useful. In [Daskalakis and Papadimitriou 2009, 196] words: "if your laptop can't find it, then, probably, neither can the market." Our results suggest that it is precisely when our laptop cannot find the equilibrium that decentralized interactions prove most useful.

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PSEUDOCODE

```
def game():
    create agents in a network
    set each player's initial state randomly
    for step in simulation length:
        players = a random subset of all players
        for each player in players:
            set the state to the most frequent state among its network neighbors
        check if all players on network converged to the same state:
            if yes: end simulation.
```

PARAMETERS

	Description	*Range*	*Value*
Model parameter			
a	Number of agents	$\mathbb{N}$	Varies
e	Number of equilibria	$\mathbb{N}$	Varies
p	Proportion of agents who are activated at every time step	$\in (0, 1)$	0.1
D	Mean degree distribution	$\mathbb{R}$	2
Data collection			
t	Maximum time steps for which the model is run	$\mathbb{N}$	100,000
f	The frequency with which the state of the system is recorded, i.e., how many time steps pass in between observations	$\mathbb{N}$	100
ϵ	$1 - \epsilon$ is the proportion of agents that must have system is considered as equilibrated	$\in (0, 1)$	10^{-3}
r	The number of times the model is run for a given model parameter value	$\mathbb{N}$	100

Notes

1. In recent years, economists have become increasingly cognizant of the role of network in generating a variety of economic phenomena, including macroeconomic volatility [Acemoglu et al. 2012], provision of public goods [Bramoullé and Kranton 2007; Szolnoki and Szabó 2009], the emergence of equilibrium prices [Wilhite 2001; Corominas-Bosch 2004], and the spread of innovation [Valente 1996]. After the recent financial crisis macroeconomists have begun to pay close attention to the topology of interaction between economic actors; for instance, [Caballero 2010, 18] says: "we need to spend much more effort in understanding the topology of interactions in real economies."
2. This paper builds on Mukherjee and Airiau [2007] work on emergence of norms on a grid, Sen and Airiau [2007] and Sen and Sen [2010] work on the emergence of norms under different learning rules on different topologies, Qin et al. [2011] study of the emergence of consensus without a leader on a directed graph, Ohtsuki et al. [2006] work on the spread of cooperation on social networks, Hong, Choi and Kim [2002] study of synchronization among coupled oscillators in a small world network (this is related to how fireflies come to flash synchronously), and Abramson and Kuperman [2001] study of prisoners' dilemma on different network topologies.
3. *NP* stands for nondeterministic polynomial time.
4. *FNP* contains the functional extensions of the decision problems of *NP*. While decision problems ask whether something exists, functional problems ask what it is. *TFNP* are those *FNP* problems for which the existence of the solution is guaranteed, which means that solutions can be searched-for knowing that they exist. *TFNP* is divided into subclasses according to the kind of mathematical proof that is used to show that a solution exists. *PPAD* is the class in which the existence proof is based on parity argument on a directed graph.
5. The class *PLS* contains problems which cannot be solved in polynomial time through local-search for optima.
6. These results provide a lower-bound of the time necessary to discover Nash equilibrium by agent-deduction. The mechanism assumes that all information necessary to solve the problem is available to the agents making the computations. In the real world, information about preferences and strategies of players is widely dispersed [Hayek 1945]. Computational complexity results assume away the time necessary to collect this information and make it available to all agents.
7. Axtell [2005] proves the stability of a k-lateral exchange process by constructing a Lyapunov function; he also presents results on the complexity of the trading algorithm. In the k-lateral exchange process, at every time step, agents form groups of size *k* and exhaust some portion of the gains from trade within their groups. This means that at each time step, the system moves *monotonically* toward equilibrium. In the *k*-lateral exchange process, local convergence means global convergence, i.e., trades within any given group always exhaust some portion of total gains from trade, thereby reducing the distance to equilibrium. This is not true for the interaction process studied in this paper, local convergence can lead to global divergence which in turn can cause local divergence, a wide variety of dynamics are possible.
8. In light of the fact that, Robert Axtell [2005] is the first economist to have proved Pareto's insight, we may call this the Pareto–Axtell Theorem.

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